

1. Find the exact values of x for which

$$\cosh 2x - 7 \sinh x = 5$$

giving your answers as natural logarithms.

(7)

$$1. \cosh 2x = \cosh^2 x + \sinh^2 x$$

$$c^2 - s^2 = 1$$

$$\therefore \cosh 2x - 7 \sinh x = 5$$

$$\therefore \cosh^2 x + \sinh^2 x - 7 \sinh x = 5$$

$$\therefore 1 + 2 \sinh^2 x - 7 \sinh x = 5$$

$$\therefore 2 \sinh^2 x - 7 \sinh x - 4 = 0$$

$$(\sinh x - 4)(2 \sinh x + 1) = 0$$

$$\therefore \sinh x = 4 \Rightarrow x = \operatorname{arsinh} 4 = \ln(4 + \sqrt{17})$$

$$\sinh x = -\frac{1}{2} \Rightarrow x = \ln \frac{\sqrt{5} - 1}{2}$$



2. The hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a and b are positive constants.

The hyperbola H has eccentricity $\frac{\sqrt{21}}{4}$ and passes through the point $(12, 5)$.

Find

(a) the value of a and the value of b ,

(4)

(b) the coordinates of the foci of H .

(1)

2.(a). @ $(12, 5)$, $\frac{144}{a^2} - \frac{25}{b^2} = 1$

Eccentricity $\Rightarrow b^2 = a^2(e^2 - 1)$

$\therefore b^2 = a^2\left(\frac{5}{16}\right)$

$\therefore b^2 = \frac{5}{16}a^2$

$\therefore \frac{144}{a^2} - \frac{25}{\frac{5}{16}a^2} = 1$

$\therefore \frac{144}{a^2} - \frac{80}{a^2} = 1$

$\therefore a^2 = 64 \Rightarrow a = 8$

$a = 8 \Rightarrow b^2 = 20 \therefore b = 2\sqrt{5}$



3.

$$M = \begin{pmatrix} 0 & 1 & 9 \\ 1 & 4 & k \\ 1 & 0 & -3 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

Given that $\begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix M ,

(a) find the eigenvalue of M corresponding to $\begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix}$, (2)

(b) show that $k = -7$ (2)

(c) find the other two eigenvalues of the matrix M . (4)

The image of the vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ under the transformation represented by M is $\begin{pmatrix} -6 \\ 21 \\ 5 \end{pmatrix}$.

(d) Find the values of the constants p , q and r . (4)

3 (a). $Mx = \lambda x$

$$\therefore \begin{pmatrix} 0 & 1 & 9 \\ 1 & 4 & k \\ 1 & 0 & -3 \end{pmatrix} \begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix} = \begin{pmatrix} 7\lambda \\ 19\lambda \\ \lambda \end{pmatrix}$$

$$\therefore \begin{pmatrix} 28 \\ 23+k \\ 4 \end{pmatrix} = \begin{pmatrix} 7\lambda \\ 19\lambda \\ \lambda \end{pmatrix}$$

$\Rightarrow \lambda = 4$ is the corresponding E. value.



Question 3 continued

$$(b) \quad \underline{m\pi} = \begin{pmatrix} 28 \\ 83+k \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix} = 4\pi$$

$$\therefore 83+k = 76$$

$$\Rightarrow k = 76 - 83$$

$$\therefore k = -7 \quad \text{as required.}$$

$$(c) \quad M - \lambda I = \begin{pmatrix} -\lambda & 1 & 9 \\ 1 & 4-\lambda & -7 \\ 1 & 0 & -3-\lambda \end{pmatrix}$$

$$\det(M - \lambda I) = 0 \Rightarrow \lambda(4-\lambda)(3+\lambda) - ((4-\lambda)(-3-\lambda))$$

$$\Rightarrow \lambda(4-\lambda)(3+\lambda) - (-3-\lambda+7) + 9(0-4+\lambda) = 0$$

$$\Rightarrow \lambda(4-\lambda)(3+\lambda) - (4-\lambda) - 9(4-\lambda) = 0$$

$$\therefore (4-\lambda) [\lambda(3+\lambda) - 10] = 0$$

$$\therefore (4-\lambda)(\lambda^2 + 3\lambda - 10) = 0$$

$$(4-\lambda)(\lambda-2)(\lambda+5) = 0$$

$$\therefore \lambda = 2 \quad \& \quad \lambda = -5$$

Question 3 continued

$$\begin{pmatrix} 0 & 1 & 9 \\ 1 & 4 & -7 \\ 1 & 0 & -3 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -6 \\ 21 \\ 5 \end{pmatrix}$$

$$\therefore \begin{pmatrix} q + 9r \\ p + 4q - 7r \\ p - 3r \end{pmatrix} = \begin{pmatrix} -6 \\ 21 \\ 5 \end{pmatrix}$$

$$\therefore \begin{aligned} q + 9r &= -6 \\ \Rightarrow r &= \frac{-6 - q}{9} \end{aligned}$$

$$\begin{aligned} \therefore \cancel{p - 3r = 5} \quad p &= 5 + 3r = 5 + \frac{6 + q}{3} \\ p &= 3 - \frac{q}{3} \end{aligned}$$

$$p + 4q - 7r = 21$$

$$\therefore 3 - \frac{q}{3} + 4q + \frac{7}{9}(6 + q) = 21$$

$$\therefore \frac{40}{9}q + \frac{23}{3} = 21$$

$$\Rightarrow q = 3$$

$$\therefore p = 3 - 1 = 2 \quad \therefore r = -1 \quad \left. \begin{array}{l} p = 2 \\ q = 3 \\ r = -1 \end{array} \right\} //$$

4. $I_n = \int \cosh^n x \, dx, \quad n \geq 0$

(a) Show that, for $n \geq 2$

$$nI_n = \sinh x \cosh^{n-1} x + (n-1)I_{n-2} \quad (6)$$

(b) Hence find the exact value of

$$\int_0^{\ln 2} \cosh^5 x \, dx \quad (4)$$

4(a) $I_n = \int \cosh^n x \, dx = \int \cosh^{n-1} x \cosh x \, dx$

Let $u = \cosh^{n-1} x \quad \therefore u' = (n-1) \cosh^{n-2} x \sinh x$
 $\therefore u' = \sinh x (n-1) \cosh^{n-2} x$

$v' = \cosh x \quad \therefore v = \sinh x$

$\therefore I_n = \sinh x \cosh^{n-1} x - (n-1) \int \sinh^2 x \cosh^{n-2} x \, dx$

Use $\sinh^2 x = \cosh^2 x - 1$

$\therefore I_n = \sinh x \cosh^{n-1} x - (n-1) \int \cosh^n x - \cosh^{n-2} x \, dx$

$\therefore I_n = \sinh x \cosh^{n-1} x - (n-1) (I_n - I_{n-2})$

$\therefore I_n = \sinh x \cosh^{n-1} x - (n-1) I_n + (n-1) I_{n-2}$

$\therefore n I_n = \sinh x \cosh^{n-1} x + (n-1) I_{n-2}$

as required.

Question 4 continued

$$(b) \quad \text{Let } I_n = \int_0^{\ln 2} \cosh^n x \, dx$$

$$n=5 \rightarrow \cancel{5} I_5$$

$$\therefore n I_n = \left[\sinh x \cosh^{n-1} x \right]_0^{\ln 2} + (n-1) I_{n-2}$$

$$\therefore n I_n = \cancel{\frac{3}{4}} \cosh^n$$

$$\therefore n I_n = \frac{3}{4} \left(\frac{5}{4}\right)^{n-1} + (n-1) I_{n-2}$$

$$n=5 \Rightarrow$$

$$5 I_5 = \frac{1875}{1024} + 4 I_3$$

$$n=3 \Rightarrow$$

$$3 I_3 = \frac{75}{64} + 2 I_1$$

$$I_1 = \int_0^{\ln 2} \cosh x \, dx = \left[\sinh x \right]_0^{\ln 2} = \frac{3}{4}$$

$$\therefore I_3 = \frac{57}{64} \Rightarrow I_5 = \frac{5523}{5120} \quad 1.078$$

5. The ellipse E has equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$

The line L has equation $y = mx + c$, where m and c are constants.

Given that L is a tangent to E ,

(a) show that

$$c^2 - 25m^2 = 9 \quad (4)$$

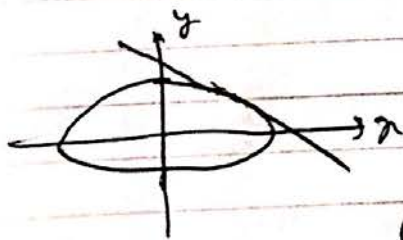
(b) find the equations of the tangents to E which pass through the point $(3, 4)$. (5)

5 (a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ & $y = mx + c$

$$\Rightarrow \frac{x^2}{25} + \frac{(mx+c)^2}{9} = 1$$

$$\therefore \frac{x^2}{25} + \frac{m^2x^2 + 2m cx + c^2}{9} = 1$$

(x225) $\Rightarrow 9x^2 + 25m^2x^2 + 50m cx + 25c^2 = 225$
 $\therefore (9 + 25m^2)x^2 + 50m cx + 25c^2 - 225 = 0$



Ellipse & L intersect once
 if L is tangent to E .

Quadratic has **only** one root.

\therefore discriminant $= 0$
 $\Rightarrow 50^2 m^2 c^2 - 4(9 + 25m^2)(25c^2 - 225) = 0$

$$2500m^2c^2 - 4(225c^2 - 2025 + 625m^2c^2 - 5625m^2) = 0$$

Question 5 continued

$$\therefore 2500 m^2 c^2 - 900 c^2 + 8100 - 2500 m^2 c^2 + 22500 m^2 = 0$$

$$\therefore -900 c^2 + 22500 m^2 + 8100 = 0$$

$$\downarrow \div (-900)$$

$$\Rightarrow c^2 - 25m^2 - 9 = 0$$

$$\Rightarrow c^2 - 25m^2 = 9$$

as required.

$$(b) (3, 4) \Rightarrow 4 = 3m + c$$

$$\Rightarrow c = 4 - 3m$$

$$c^2 - 25m^2 = 9$$

$$\therefore (4 - 3m)^2 - 25m^2 = 9$$

$$\therefore 16 - 24m + 9m^2 - 25m^2 - 9 = 0$$

$$\therefore 16m^2 + 24m - 7 = 0$$

$$\Rightarrow (4m - 1)(4m + 7) = 0$$

$$\Rightarrow m = \frac{1}{4} \Rightarrow c = \frac{13}{4}$$

$$m = -\frac{7}{4} \Rightarrow c = \frac{37}{4}$$

$$\therefore y = \frac{1}{4}x + \frac{13}{4} \quad \& \quad y = -\frac{7}{4}x + \frac{37}{4}$$

6.

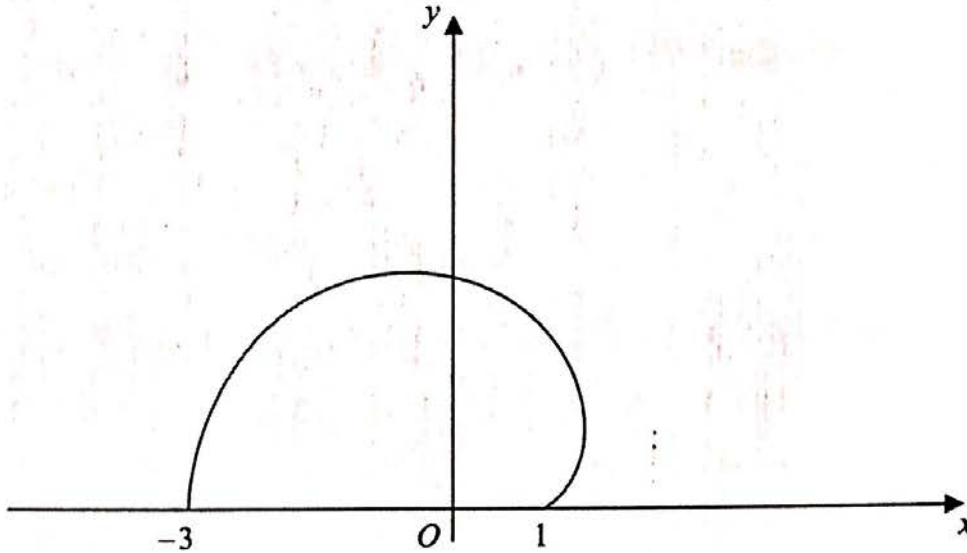


Figure 1

Figure 1 shows the curve C with parametric equations

$$x = 2\cos\theta - \cos 2\theta, \quad y = 2\sin\theta - \sin 2\theta, \quad 0 \leq \theta \leq \pi$$

(a) Show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 8(1 - \cos\theta) \quad (5)$$

The curve C is rotated through 2π radians about the x -axis.

(b) Find the area of the surface generated, giving your answer in the form $k\pi$, where k is a rational number.

(5)

$$x = 2\cos\theta - \cos 2\theta$$

$$\therefore \frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$$

$$y = 2\sin\theta - \sin 2\theta$$

$$\frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$$



$$\left(\frac{dx}{d\theta}\right)^2 = (2\sin 2\theta - 2\sin\theta)^2 = 4\sin^2 2\theta - 8\sin 2\theta \sin\theta + 4\sin^2\theta$$

$$\left(\frac{dy}{d\theta}\right)^2 = (2\cos\theta - 2\cos 2\theta)^2 = 4\cos^2\theta - 8\cos 2\theta \cos\theta + 4\cos^2 2\theta$$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 =$$

$$4\sin^2 2\theta - 8\sin 2\theta \sin\theta + 4\sin^2\theta + 4\cos^2\theta - 8\cos 2\theta \cos\theta + 4\cos^2 2\theta$$

$$= 4(\sin^2 2\theta + \cos^2 2\theta) + 4(\sin^2\theta + \cos^2\theta) - 8\sin 2\theta \sin\theta - 8\cos 2\theta \cos\theta$$

$$= \underline{\underline{8}} - 8(\sin 2\theta \sin\theta)$$

$$= \underline{\underline{8}} - 8(\cos 2\theta \cos\theta + \sin 2\theta \sin\theta)$$

Use $\cos(A-B) = \cos A \cos B + \sin A \sin B$
 $A = 2\theta \quad B = \theta$

$$\therefore = 8 - 8(\cos(2\theta - \theta)) = 8 - 8\cos\theta$$
$$= 8(1 - \cos\theta)$$

as required.

$$S = 2\pi \int_0^{\pi} (2s\sin\theta - \sin^2\theta) \sqrt{8(1-\cos\theta)} d\theta$$

$$= 4\sqrt{2}\pi \int_0^{\pi} 2s\sin\theta(1-\cos\theta) \sqrt{1-\cos\theta} d\theta$$

$$= 8\sqrt{2}\pi \int_0^{\pi} \sin\theta(1-\cos\theta)^{3/2} d\theta$$

$$= 8\sqrt{2}\pi \left[\frac{2}{5}(1-\cos\theta)^{5/2} \right]_0^{\pi}$$

$$= 8\sqrt{2}\pi \left(\frac{2}{5}(2)^{5/2} \right) \quad 4\sqrt{2}$$

$$= 8\sqrt{2}\pi \left(\frac{2}{5}(\sqrt{2})^5 \right)$$

$$= 8\sqrt{2}\pi \times \frac{8}{5}\sqrt{2}$$

$$= \frac{128}{5}\pi$$

~~ans~~
 $\mu = \frac{128}{5}$



7. The plane Π_1 contains the point $(3, 3, -2)$ and the line $\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z+1}{4}$

(a) Show that a cartesian equation of the plane Π_1 is

$$3x - 10y - 4z = -13 \quad (5)$$

The plane Π_2 is parallel to the plane Π_1

The point $(\alpha, 1, 1)$, where α is a constant, lies in Π_2

Given that the shortest distance between the planes Π_1 and Π_2 is $\frac{1}{\sqrt{5}}$

(b) find the possible values of α . (6)

$$7(a). \quad x-1 = 4 - 2y = \frac{2z+2}{4}$$

$$\therefore 4x-4 = 16-8y = 2z+2$$

$$4x-4 = 16-8y$$

$$\therefore x = \frac{20-8y}{4} = 5-2y \quad y=y$$

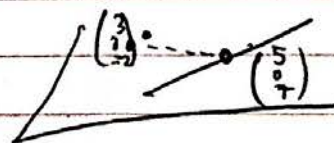
$$\text{or } 2z+2 = 16-8y$$

$$\therefore z = 7-4y$$

$$\text{let } y = \lambda$$

line has eqn $\underline{r} = \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$

$$\begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$$



$$\begin{matrix} 2 & -3 & 9 \\ -2 & 1 & -4 \end{matrix} \times \begin{matrix} 2 & -3 & 9 \\ -2 & 1 & -4 \end{matrix}$$

Question 7 continued

Limit

$$\therefore \eta = \begin{pmatrix} 2 \\ -3 \\ 9 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -10 \\ -4 \end{pmatrix}$$

$$\underline{r} \cdot \underline{\eta} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -10 \\ -4 \end{pmatrix}$$

$$\Rightarrow \underline{r} \cdot \underline{\eta} = 9 - 30 + 8 = -13$$

$$\therefore \underline{r} \cdot \begin{pmatrix} 3 \\ -10 \\ -4 \end{pmatrix} = -13$$

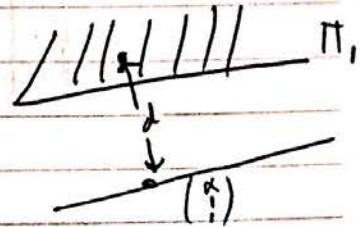
$$\Rightarrow 3x - 10y - 4z = -13 \quad \text{as required.}$$

(b) Π_2 contains line: $\vec{r} = \begin{pmatrix} \alpha \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} \dots \\ \dots \\ \dots \end{pmatrix}$

(b) Shortest distance = perp. distance from $\begin{pmatrix} \alpha \\ 1 \\ 1 \end{pmatrix}$ to Π_1

Use formula in booklet with

~~3x~~ $3x - 10y - 4z + 13 = 0$



$$d = \frac{|3\alpha - 10 - 4 + 13|}{\sqrt{3^2 + 10^2 + 4^2}} = \frac{|3\alpha - 1|}{5\sqrt{5}}$$

$$\Rightarrow \frac{|3\alpha - 1|}{5\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\Rightarrow |3\alpha - 1| = 5$$

$$\therefore 3\alpha - 1 = 5 \Rightarrow \alpha = 2$$

$$1 - 3\alpha = 5 \Rightarrow \alpha = -\frac{4}{3}$$



8. (a) Show that, under the substitution $x = \frac{3}{4} \sinh u$,

$$\int \frac{x^2}{\sqrt{16x^2 + 9}} dx = k \int (\cosh 2u - 1) du$$

where k is a constant to be determined.

(6)

(b) Hence show that

$$\int_0^1 \frac{64x^2}{\sqrt{16x^2 + 9}} dx = p + q \ln 3$$

where p and q are rational numbers to be found.

(5)

$$8(a) \quad x = \frac{3}{4} \sinh u \quad \therefore \frac{dx}{du} = \frac{3}{4} \cosh u$$

$$\therefore dx = \frac{3}{4} \cosh u \, du$$

$$c^2 - s^2 = 1$$

$$\& \quad \frac{x^2}{\sqrt{16x^2 + 9}} = \frac{\frac{9}{16} \sinh^2 u}{\sqrt{9 \sinh^2 u + 9}} = \frac{9/16 \sinh^2 u}{3 \cosh u}$$

$$= \frac{3}{16} \frac{\sinh^2 u}{\cosh u}$$

$$\therefore \int \frac{x^2}{\sqrt{16x^2 + 9}} dx = \frac{3}{16} \int \frac{\sinh^2 u}{\cosh u} \cdot \frac{3}{4} \cosh u \, du$$

$$= \frac{9}{64} \int \sinh^2 u \, du$$



$$= \frac{9}{128} \int \cosh 2u - 1 \, du$$

$u = \frac{9}{128}$

$$(b) \quad 1 = \frac{3}{4} \sinh ku \Rightarrow u = \operatorname{arsinh} \frac{4}{3} = \ln 3$$

$$0 = \sinh ku \Rightarrow u = 0$$

$$\therefore \frac{9}{2} \int_0^{\ln 3} \cosh 2u - 1 \, du$$

$$= \frac{9}{2} \left[\frac{1}{2} \sinh 2u - u \right]_0^{\ln 3}$$

$$= \frac{9}{2} \left(\frac{1}{2} \sinh \ln 9 - \ln 3 \right)$$

$$= \frac{9}{2} \left(\frac{1}{4} (9 - 9^{-1}) - \ln 3 \right)$$

$$= 10 - \frac{9}{2} \ln 3$$